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The Approximation of Cauchy-Type Integrals by Some Kinds of Interpolatory Splines

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1. INTRODUCTION

There are many references on numerical evaluation of Cauchy-type integrals

$$T_L f(t) = \frac{1}{\pi i} \int_L \frac{f(\tau)}{\tau - t}\, d\tau, \qquad t \in L \tag{1.1}$$

(possibly, with a weight function), using orthogonal polynomials when L is an interval on the real axis, e.g., [1–4]. If L is the unit circle, it was shown in [5] that such integrals are approximated by interpolatory polynomial splines of any odd degree under the assumption that the density function $f(t)$ is holomorphic in the interior and continuous on the boundary of the circle up to a certain order of its derivatives. In [6], the same problem under a similar hypothesis on $f(t)$ was discussed for cubic interpolating splines in case L is an arbitrary smooth closed contour.

Let L be an arbitrary smooth curve, closed or open ($L = \widehat{ab}$), and

$$\Delta\colon a = t_0 \prec t_1 \prec \cdots \prec t_N = b$$

be a partition of L ($t_N = t_0$ when L is closed), where $L_j \prec L_{j+1}$ means that L_j precedes L_{j+1} when one travels along L in its given direction. If $f(t)$ is a function $\in H^\alpha$ (Hölder condition) on L and $S_\Delta(t)$ linearly interpolates $f(t)$ at the t_j's, Atkinson [7] succeeded in proving the uniform convergence of $T_L S_\Delta$ to $T_L f$ when L is closed,

$$\| T_L f - T_L S_\Delta \|_\infty \leqslant C_\varepsilon \delta^{\alpha - \varepsilon}, \qquad \delta = \max |t_{j+1} - t_j|, \tag{1.2}$$

where C_ε is a constant independent of Δ, provided

$$K_\Delta = \max |t_{j+1} - t_j| / \min |t_{j+1} - t_j| \tag{1.3}$$

is bounded for all the Δ's.

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This result was proved by using a theorem (cf. [7, Theorem 2]) which itself depends on an interesting but complicated lemma. We find that it may be easily proved by the following simple approach. Let L be a smooth contour and $f(t) \in H^\alpha$ $(0 < \alpha < 1)$ on it. In the Banach space H^α, we know [8] that

$$\|f\|_{H^\alpha} = \|f\|_\infty + M_\alpha(f), \tag{1.4}$$

where

$$M_\alpha(f) = \sup_{t,t' \in L} \frac{|f(t) - f(t')|}{|t - t'|^\alpha}; \tag{1.5}$$

then the operator T_L is linear with norm $\|T_L\|_\alpha$. Therefore,

$$\|T_L f\|_\infty \leqslant \|T_L f\|_{H^\alpha} \leqslant \|T_L\|_\alpha \|f\|_{H^\alpha} = \|T_L\|_\alpha [\|f\|_\infty + M_\alpha(f)]. \tag{1.6}$$

Now, if $f \in H^\varepsilon$ $(0 < \varepsilon < 1)$ and $f_n \in H^\varepsilon$ is a sequence of functions on L, and if we can estimate $\|e_n(t)\|_\infty = \|f(t) - f_n(t)\|_\infty$ and $M_\varepsilon(e_n)$, then, by (1.6), we may estimate $\|T_L e_n\|_\infty$ by

$$\|T_L e_n\|_\infty \leqslant C_\varepsilon[\|e_n\|_\infty + M_\varepsilon(e_n)].^{1} \tag{1.7}$$

We shall use (1.7) to estimate $\|T_L e_\Delta\|_\infty = \|T_L f - T_L S_\Delta\|_\infty$ both in the cases $f(t) \in H^\alpha$ and $f(t) \in C^1$.

The structure of $T_L S_\Delta(t)$ is very simple because of its linearity on each sub-arc of L. However, $S_\Delta(t)$, as well as $T_L S_\Delta(t)$, is not smooth in general even if $f(t)$ is smooth. We shall establish analogous results for quadratic interpolating splines[2] in place of linear ones, so that $T_L S_\Delta(t)$ will be smooth. However, if $f'(t) \in H^\alpha$ on L, we could not conclude the convergence of $T_L S'_\Delta(t)$ to $T_L f'(t)$. Using cubic interpolating splines of deficiency 2 which were discussed in [9], we may establish such convergence. We also establish the convergence of $T_L S^*_\Delta(t)$ to $T_L f(t)$ when $f \in H^\alpha$, where S^*_Δ is the modified cubic interpolating spline of deficiency 2 which was also introduced in [9]. Here $T_L S^*_\Delta(t)$ as well as $S^*_\Delta(t)$ is smooth.

The results of this paper are also valid when L is an open smooth arc $\widehat{ab}$.

[1] We use the symbol C_ε to represent a constant depending on ε which may take different values in different cases. Similarly, C represents an absolute constant taking various values in various cases.

[2] K. Atkinson also proved such convergence for "quadratic interpolating splines" which are different in meaning from those introduced here. He made interpolations of the values of $f(t)$ at three consecutive knots by a quadratic function. Such splines, in general, are not smooth too.

To show this, we note that (1.7) remains true in this case, provided that the additional requirements

$$f_n(a)=f(a), \qquad f_n(b)=f(b) \tag{1.8}$$

are fulfilled. In fact, we may extend L to a smooth contour L^* and simultaneously extend $f(t)$ to $f^*(t)$ on L^* such that $f^*(t)\in H^\varepsilon$ on L^*. Since (1.8) is satisfied, we may extend $f_n(t)$ to $f_n^*(t)$ on L^* such that $f_n^*(t)\equiv f^*(t)$ on L^*-L, so that $f_n^*(t)\in H^\varepsilon$ on L^* too. Let $e_n^*=f^*-f_n^*$; then (1.7) is valid for e_n^*. Now $T_{L^*}e_n^*=T_L e_n$, $\|e_n\|_\infty=\|e_n^*\|_\infty$. Let us estimate $M_\varepsilon(e_n^*)$. If $t, t'\in L$, then

$$|e_n^*(t)-e_n^*(t')|\leqslant M_\varepsilon(e_n)\,|t-t'|^\varepsilon;$$

if $t, t'\in L^*-L$, this is trivial. Let $t\in L$, $t'\in L^*-L$ and let a be situated on the shorter arc of $\widehat{tt'}$. Then

$$\begin{aligned}|e_n^*(t)-e_n^*(t')|&=|e_n(t)|=|e_n(t)-e_n(a)|\leqslant M_\varepsilon(e_n)\,|t-a|^\varepsilon\\ &\leqslant M_\varepsilon(e_n)\,|\widehat{tt'}|^\varepsilon\leqslant C_\varepsilon M_\varepsilon(e_n)\,|t-t'|^\varepsilon,\end{aligned}$$

where we have used a well-known inequality

$$|\widehat{tt'}|\leqslant C\,|t-t'|. \tag{1.9}$$

Therefore,

$$M_\varepsilon(e_n^*)\leqslant C_\varepsilon M_\varepsilon(e_n)$$

and thence (1.7) remains valid.

2. The Linear Interpolating Splines

Let L be a smooth contour and $f(t)\in H^\alpha$ $(0<\alpha<1)$ on it. Denote

$$L_j=\widehat{t_j t_{j+1}}, \qquad y_j=f(t_j), \qquad \Delta y_j=y_{j+1}-y_j, \qquad D_j=\Delta y_j/\Delta t_j.$$

(We use the conventions $y_{j+N}=y_j$, $\Delta y_{j+n}=\Delta y_j$, etc.) Then, for any linear interpolating spline $S_\Delta(t)$, we have

$$S_\Delta(t)\equiv S_j(t)=y_j+D_j(t-t_j), \qquad t\in L_j, \quad j=0,1,\dots,N-1. \tag{2.1}$$

It is evident that

$$|e_\Delta(t)|=|f(t)-S_\Delta(t)|\leqslant C\delta^\alpha \tag{2.2}$$

since, by (1.9),

$$|D_j(t-t_j)| \leqslant C\delta^\alpha \frac{|t-t_j|}{|\Delta t_j|} \leqslant C\delta^\alpha \frac{\Delta s_j}{|\Delta t_j|} \leqslant C\delta^\alpha, \qquad t \in L_j,$$

where Δs_j is the arc-length of L_j. Similarly, if t, t' belong to the same L_j, then

$$|e_\Delta(t) - e_\Delta(t')| \leqslant C|t-t'|^\alpha \leqslant C\delta^{\alpha-\varepsilon}|t-t'|^\varepsilon. \tag{2.3}$$

If $t \in L_j$, $t' \in L_k$, $j \neq k$, then

$$\begin{aligned}|e_\Delta(t) - e_\Delta(t')| &\leqslant |e_\Delta(t)| + |e_\Delta(t')| = |e_\Delta(t) - e_\Delta(t_j)| + |e_\Delta(t_k) - e_\Delta(t')| \\ &\leqslant C[|t-t_j|^\alpha + |t_k - t'|^\alpha] \leqslant C\delta^{\alpha-\varepsilon}[|t-t_j|^\varepsilon + |t_k - t'|^\varepsilon] \\ &\leqslant C\delta^{\alpha-\varepsilon}(|\widehat{tt_j}|^\varepsilon + |\widehat{t_k t'}|^\varepsilon \leqslant C\delta^{\alpha-\varepsilon}|\widehat{tt'}|^\varepsilon \\ &\leqslant C\delta^{\alpha-\varepsilon}|t-t'|^\varepsilon, \end{aligned} \tag{2.3)'$$

i.e., (2.3) remains true. Therefore

$$M_\varepsilon(e_\Delta) \leqslant C\delta^{\alpha-\varepsilon}. \tag{2.4}$$

From (2.2) and (2.4), we obtain, by (1.7),

$$\|T_L e_\Delta\|_\infty \leqslant C_\varepsilon \delta^{\alpha-\varepsilon}. \tag{2.5}$$

Obviously, it is then also true for $\alpha = 1$.

Now, let us consider the case $f(t) \in C^1$. We denote the modulus of continuity of $f^{(r)}(t)$ by $\omega_r(\delta)$ $(r \geqslant 0)$ throughout the paper. Then, if $t \in L_j$,

$$\begin{aligned}|f(t) - y_j - D_j(t-t_j)| &= \left| \int_{t_j}^t \left[f'(\tau) - D_j \right] d\tau \right| \\ &= \left| \frac{1}{\Delta t_j} \int_{t_j}^t d\tau \int_{t_j}^{t_{j+1}} [f'(\tau) - f'(\zeta)]\, d\zeta \right| \\ &\leqslant \frac{\omega_1(\delta)}{|\Delta t_j|} \Delta s_j^2 \leqslant C\omega_1(\delta)\,\delta. \end{aligned} \tag{2.6}$$

Similarly, if $t, t' \in L_j$, we have

$$|e_\Delta(t) - e_\Delta(t')| \leqslant C\omega_1(\delta)\,|t-t'| \leqslant C\omega_1(\delta)\,\delta^{1-\varepsilon}|t-t'|^\varepsilon; \tag{2.7}$$

if t, t' belong to different L_j's, then as in (2.3)$'$, we also get (2.7). Hence

$$M_\varepsilon(e_\Delta) \leqslant C\omega_1(\delta)\,\delta^{1-\varepsilon}. \tag{2.8}$$

Again by (1.7), we have from (2.6) and (2.8),

$$\|T_L e_\Delta\|_\infty \leqslant C_\varepsilon \omega_1(\delta)\, \delta^{1-\varepsilon}. \tag{2.9}$$

If $L = \widehat{ab}$ is an open smooth arc, since

$$S_\Delta(a) = f(a), \qquad S_\Delta(b) = f(b), \tag{2.10}$$

(2.5) and (2.9) remain valid.

Thus, we obtain

THEOREM 1. *Let L be a smooth curve, closed or open, and $S_\Delta(t)$ be the linear interpolating spline of $f(t)$. If $f(t) \in H^\alpha$ $(0 < \alpha \leqslant 1)$, then*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \delta^{\alpha-\varepsilon};$$

if $f(t) \in C^1$, then

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \omega_1(\delta)\, \delta^{1-\varepsilon}.$$

COROLLARY 1. *If $f'(t) \in H^\alpha$ $(0 < \alpha \leqslant 1)$, then*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \delta^{1+\alpha-\varepsilon}. \tag{2.11}$$

COROLLARY 2. *If $|f''(t)|$ is bounded, then*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \delta^{2-\varepsilon}. \tag{2.12}$$

Corollary 2 is also a result due to Atkinson for closed L.

We note that all the values of the C_ε's in this section do not depend on Δ and so are independent of K_Δ in (1.3); therefore it is not necessary to require K_Δ to be bounded as stated in [7].

3. THE QUADRATIC INTERPOLATING SPLINES

Though there are works describing briefly polynomial interpolating splines on a Jordan curve [10, 11] and dealing with quadratic splines on an interval of the real axis [12, 13], we shall discuss the latter in the complex domain somewhat in detail, whether L is closed or open.

First, let us consider the case L is closed. A quadratic spline $S_\Delta(t)$ interpolating $f(t)$ at t_j's, if any, may be represented in various ways, for instance,

$$S_\Delta(t) \equiv S_j(t) = y_j + D_j(t - t_j) + \frac{A_j}{\Delta t_j}(t - t_j)(t_{j+1} - t),$$

$$t \in L_j, \quad j = 0, 1, \ldots, N-1, \tag{3.1}$$

with the requirements

$$A_{j-1} + A_j = -\Delta D_{j-1}, \qquad j = 1,..., N, \tag{3.2}$$

where $\Delta D_{j-1} = D_j - D_{j-1}$, so as to guarantee the continuity of $S'_\Delta(t)$ at $t = t_j$.

If N is odd, we readily see that (3.2) is uniquely solvable:

$$\begin{aligned} A_j &= -\tfrac{1}{2}(\Delta D_j - \Delta D_{j+1} + \Delta D_{j+2} - \Delta D_{j+3} + \cdots + \Delta D_{j+N-1}) \\ &= \Delta D_{j+1} + \Delta D_{j+3} + \cdots + \Delta D_{j+N-2}, \qquad j = 0, 1,..., N-1, \end{aligned} \tag{3.3}$$

since $\sum_{j=0}^{N-1} D_j = 0$.

If N is even, (3.2) is solvable iff

$$\Delta D_0 + \Delta D_2 + \cdots + \Delta D_{N-2} = 0 \tag{3.4}$$

or

$$D_0 + D_2 + \cdots + D_{N-2} = D_1 + D_3 + \cdots + D_{N-1}. \tag{3.4$'$}$$

In the case $L = \widehat{ab}$ is an open arc, then, for such a spline, expression (3.1) remains effective, but requirements (3.2) are replaced by

$$A_{j-1} + A_j = -\Delta D_{j-1}, \qquad j = 1,..., N-1. \tag{3.5}$$

Hence, we have a freedom to choose A_0 or A_n. Or, more generally, we may subject them to an additional relation

$$\alpha A_0 + \beta A_{N-1} = \gamma, \qquad \beta \neq (-1)^N \alpha. \tag{3.6}$$

On solving (3.5) and (3.6), we get

$$A_0 = \frac{(-1)^N \gamma + B_{N-2}}{(-1)^N \alpha - \beta}, \qquad A_j = (-1)^j (A_0 + B_{j-1}), \qquad j = 1,..., N-1, \tag{3.7}$$

where

$$B_j = \Delta D_0 - \Delta D_1 + \Delta D_2 - \Delta D_3 + \cdots + (-1)^j \Delta D_j, \quad j = 0, 1,..., N-2. \tag{3.8}$$

Thus, we have

THEOREM 2. *If L is closed, the quadratic interpolating spline $S_\Delta(t)$ exists uniquely when N is odd and it exists (but not uniquely) iff* (3.4) *or* (3.4)$'$ *is fulfilled when N is even; if L is open, it exists uniquely for arbitrary N with additional requirement* (3.6).

Now we turn to the problems of approximation.

Again we consider first the case L is smooth and closed (N: odd). We assume $f(t) \in C^1$. In order to estimate $e_\Delta(t) = f(t) - S_\Delta(t)$, by (3.1), it is necessary to estimate A_j in (3.3). Noting that

$$|\Delta D_{j-1}| \leqslant |D_j - y_j'| + |y_j' - D_{j-1}| \leqslant C\omega_1(\delta) \qquad (y_j^{(r)} = f^{(r)}(t_j))$$

we have

$$|A_j| \leqslant (N-1)\, C\omega_1(\delta), \tag{3.9}$$

and then

$$\left| \frac{A_j}{\Delta t_j} (t - t_j)(t_{j+1} - t) \right| \leqslant (N-1)\, C\omega_1(\delta)\, \delta \leqslant CK_\Delta\, \omega_1(\delta), \tag{3.10}$$

where K_Δ is given by (1.3). Thus, by (3.1), we obtain from (2.6) and (3.10),

$$\| e_\Delta(t) \|_\infty \leqslant CK_\Delta\, \omega_1(\delta). \tag{3.11}$$

If L is open, the similar estimate (3.9) is valid for $|B_j|$ by (3.8) and thereby also for $|A_j|$ on account of (3.7). Hence (3.10) as well as (3.11) remains true.

Therefore, we have

THEOREM 3. *For a quadratic interpolating spline $S_\Delta(t)$, if $f(t) \in C^1$, we have the estimate*

$$|f(t) - S_\Delta(t)| \leqslant CK_\Delta\, \omega_1(\delta),$$

whether L is closed (N: odd), or open (N: arbitrary) with requirement (3.6).

We could not expect $S_\Delta'(t)$ to tend to $f'(t)$ in this case even if L is closed. In fact, we can only easily obtain the estimate

$$|f'(t) - S_\Delta'(t)| \leqslant CK_\Delta \frac{\omega_1(\delta)}{\delta}. \tag{3.12}$$

Similarly, if $f(t) \in C$, we can only obtain

$$|f(t) - S_\Delta(t)| \leqslant CK_\Delta \frac{\omega(\delta)}{\delta}. \tag{3.13}$$

To estimate $\| T_L e_\Delta(t) \|_\infty$, we assume $f'(t) \in H^\alpha$ $(0 < \alpha < 1)$. Then (3.11) becomes

$$\| e_\Delta(t) \|_\infty \leqslant CK_\Delta\, \delta^\alpha. \tag{3.14}$$

If $t, t' \in L_j$, then, by (3.9),

$$
\begin{aligned}
&|e_\Delta(t) - e_\Delta(t')| \\
&\leqslant |f(t) - f(t')| + |D_j(t - t')| + \left| \frac{A_j}{\Delta t_j} (t + t' - t_j - t_{j+1})(t - t') \right| \\
&\leqslant C\,|t - t'|^\alpha + (N-1)\, C\delta^\alpha\, |t - t'| \\
&\leqslant C\delta^{\alpha-\varepsilon} |t - t'|^\varepsilon + CK_\Delta \delta^{\alpha-\varepsilon} |t - t'|^\varepsilon \leqslant CK_\Delta\, \delta^{\alpha-\varepsilon} |t - t'|^\varepsilon. \qquad (3.15)
\end{aligned}
$$

If t, t' belong to different L_j's, we may proceed as in (2.3)′ and verify (3.15) remains true. Therefore,

$$M_\varepsilon(e_\Delta) \leqslant CK_\Delta\, \delta^{\alpha-\varepsilon}. \tag{3.16}$$

Together with (3.14), we have, by (1.7),

$$\| T_L e_\Delta \|_\infty \leqslant C_\varepsilon K_\Delta\, \delta^{\alpha-\varepsilon}. \tag{3.17}$$

Obviously, (3.17) remains valid then if $\alpha = 1$.

For open arc L, since (2.10) is fulfilled for $S_\Delta(t)$, (3.17) is also valid.

Thus we obtain

THEOREM 4. *For a quadratic interpolating spline $S_\Delta(t)$, and $f'(t) \in H^\alpha$ $(0 < \alpha \leqslant 1)$, we have the estimate*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon K_\Delta \delta^{\alpha-\varepsilon},$$

whether L is closed (N: odd), or open (N: arbitrary) with the additional requirement (3.6).

When $K_\Delta < C$ for a set of $\{\Delta\}$, (3.11) and (3.17) mean the corresponding uniform convergency when $\delta = \max |\Delta t_j| \to 0$.

4. THE CUBIC INTERPOLATING SPLINES OF DEFICIENCY 2

Let L be a smooth curve, closed or not. The cubic interpolating spline of deficiency 2 may be represented as

$$
\begin{aligned}
S_\Delta(t) \equiv S_j(t) = y_j + D_j(t - t_j) + \frac{y'_j - D_j}{\Delta t_j^2} (t - t_j)(t - t_{j+1})^2 \qquad (4.1) \\
+ \frac{y'_{j+1} - D_j}{\Delta t_j^2} (t - t_j)^2 (t - t_{j+1}), \qquad t \in L_j, \quad j = 0, \ldots, N-1.
\end{aligned}
$$

We proved in [9]: if $f(t) \in C^r$ $(r = 1, 2, 3)$, then

$$|e_\Delta^{(p)}(t)| = |f^{(p)}(t) - S_\Delta^{(p)}(t)| \leqslant C\omega_r(\delta)\, \delta^{r-p} \qquad (0 \leqslant p \leqslant r). \tag{4.2}$$

Let us estimate $\| T_L e_\Delta^{(p)} \|_\infty$, $p = 0, 1$. We could not expect to estimate it for $p = 2, 3$, since $T_L S_\Delta^{(p)}(t)$ have unbounded discontinuities at the knots t_j's in these cases.

First, we consider L as closed. We assume $f(t) \in C^1$. If $t, t' \in L_j$, then by (4.1),

$$\begin{aligned} e_\Delta(t) - e_\Delta(t') &= [f(t) - f(t') - D_j(t - t')] \\ &\quad + \frac{y_j' - D_j}{\Delta t_j^2} [(t - t_j)(t - t_{j+1})^2 - (t' - t_j)(t' - t_{j+1})^2] \\ &\quad + \frac{y_{j+1}' - D_j}{\Delta t_j^2} [(t - t_j)^2 (t - t_{j+1}) - (t' - t_j)^2 (t' - t_{j+1})] \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.3}$$

Analogous to (2.6), we have

$$|I_1| \leqslant C\omega_1(\delta)\, |t - t'|.$$

Noting that

$$\begin{aligned} &|(t - t_j)(t - t_{j+1})^2 - (t' - t_j)(t' - t_{j+1})^2| \\ &\quad = \left| \int_{t'}^{t} \frac{d}{d\tau} [(\tau - t_j)(\tau - t_{j+1})^2]\, d\tau \right| \\ &\quad = \left| \int_{t'}^{t} (\tau - t_{j+1})(3\tau - 2t_j - t_{j+1})\, d\tau \right| \\ &\quad \leqslant \left| \int_{t'}^{t} (s_{j+1} - s)(s + s_{j+1} - 2s_j)\, ds \right| \leqslant C\, \Delta s_j^2\, |t - t'|, \end{aligned}$$

where s_j is the arc-length coordinate of t_j, we have

$$\begin{aligned} |I_2| &\leqslant \left| \frac{1}{\Delta t_j^2} \int_{t'}^{t} [f'(t_j) - f'(\tau)]\, d\tau \right| \cdot C\, \Delta s_j^2\, |t - t'| \\ &\leqslant C\omega_1(\delta)\, |t - t'| \end{aligned}$$

and a similar estimate for $|I_3|$. Therefore,

$$|e_\Delta(t) - e_\Delta(t')| \leqslant C\omega_1(\delta)\, |t - t'| \leqslant C\omega_1(\delta)\, \delta^{1-\varepsilon} |t - t'|^\varepsilon.$$

If t, t' belong to different L_j's, it is easy to prove this estimate remains true by similar reasoning as before. Hence,

$$M_\varepsilon(e_\Delta) \leqslant C\omega_1(\delta)\,\delta^{1-\varepsilon}.$$

Together with (4.2) ($r=1$, $p=0$), we obtain, by (1.7),

$$\|T_L e_\Delta\|_\infty \leqslant C_\varepsilon \omega_1(\delta)\,\delta^{1-\varepsilon}. \tag{4.4}$$

In order to estimate $\|T_L e'_\Delta\|_\infty$, we assume $f'(t) \in H^\alpha$ $(0 < \alpha < 1)$. Since

$$S'_\Delta(t) = D_j + \frac{y'_j - D_j}{\Delta t_j^2}(t - t_{j+1})(3t - 2t_j - t_{j+1})$$
$$+ \frac{y'_{j+1} - D_j}{\Delta t_j^2}(t - t_j)(3t - t_j - 2t_{j+1}), \qquad t \in L_j,$$

if $t, t' \in L_j$,

$$e'_\Delta(t) - e'_\Delta(t')$$
$$= [f'(t) - f'(t')] + \frac{y'_j - D_j}{\Delta t_j^2}\int_{t'}^{t} \frac{d}{d\tau}(\tau - t_{j+1})(3\tau - 2t_j - t_{j+1})\,d\tau$$
$$+ \frac{y'_{j+1} - D_j}{\Delta t_j^2}\int_{t'}^{t} \frac{d}{d\tau}(\tau - t_j)(3\tau - t_j - 2t_{j+1})\,d\tau = J_1 + J_2 + J_3.$$

We have

$$|J_2| \leqslant C\frac{\delta^\alpha}{|\Delta t_j|^2}\left|\int_{t'}^{t}(3\tau - 2t_{j+1} - t_j)\,d\tau\right|$$
$$\leqslant C\frac{\delta^\alpha}{|\Delta t_j|^2}\left|\int_{t'}^{t}(2s_1 - s_0 - s)\,ds\right|$$
$$\leqslant C\frac{\delta^\alpha}{|\Delta t_j|}|t - t'| \leqslant C\delta^{\alpha-\varepsilon}|t - t'|^\varepsilon K_\Delta^\varepsilon$$

and a similar estimate for $|J_3|$. Obviously, this is also true for $|J_1|$. Therefore,

$$|e'_\Delta(t) - e'_\Delta(t')| \leqslant C\delta^{\alpha-\varepsilon}|t - t'|^\varepsilon K_\Delta^\varepsilon.$$

which remains true if t, t' belong to different L_j's. Thus,

$$M_\varepsilon(e'_\Delta) \leqslant C\delta^{\alpha-\varepsilon}K_\Delta^\varepsilon.$$

By virtue of (4.2), we have, by (1.7),

$$\| T_L e'_\Delta \|_\infty \leqslant C_\varepsilon \delta^{\alpha-\varepsilon} K_\Delta^\varepsilon. \tag{4.5}$$

Then, obviously, it remains true for $\alpha = 1$.

Let us now assume $f(t) \in C^2$. We may obtain a better estimate for $\| T_L e_\Delta \|_\infty$ as well as $\| T_L e'_\Delta \|_\infty$. We rewrite $S_j(t)$ as [9]

$$\begin{aligned} S_j(t) &= y_j + y'_j(t-t_j) + \frac{1}{2} y''_j (t-t_j)^2 \\ &\quad - \left[\Delta y_j - y'_j \Delta t_j - \frac{1}{2} y''_j \Delta t_j^2 \right] \frac{(t-t_j)^2 (t + t_j - 2t_{j+1})}{\Delta t_j^3} \\ &\quad - \left[\Delta y_j - y'_{j+1} \Delta t_j + \frac{1}{2} y''_{j+1} \Delta t_j^2 \right] \frac{(t-t_j)^2 (t - t_{j+1})}{\Delta t_j^3} \\ &\quad + \frac{y''_{j+1} - y''_j}{2} \frac{(t-t_j)^2 (t-t_{j+1})}{\Delta t_j}. \end{aligned} \tag{4.6}$$

If $t, t' \in L_j$, we have

$$\begin{aligned} e_\Delta(t) - e_\Delta(t') &= \left\{ f(t) - f(t') - y'_j(t-t') - \frac{1}{2} y''_j [(t-t_j)^2 - (t'-t_j)^2] \right\} \\ &\quad - \left[\Delta y_j - y'_j \Delta t_j - \frac{1}{2} y''_j \Delta t_j^2 \right] \\ &\quad \times \frac{1}{\Delta t_j^3} \int_{t'}^{t} \frac{d}{d\tau} [(\tau - t_j)^2 (\tau + t_j - 2t_{j+1})] \, d\tau \\ &\quad - \left[\Delta y_j - y'_{j+1} \Delta t_j + \frac{1}{2} y''_{j+1} \Delta t_j^2 \right] \\ &\quad \times \frac{1}{\Delta t_j^3} \int_{t'}^{t} \frac{d}{d\tau} [(\tau - t_j)^2 (\tau - t_{j+1})] \, d\tau \\ &\quad - \frac{y''_{j+1} - y''_j}{2\Delta t_j} \int_{t'}^{t} \frac{d}{d\tau} [(\tau - t_j)^2 (\tau - t_{j+1})] \, d\tau \\ &= H_1 + H_2 + H_3 + H_4. \end{aligned}$$

We have

$$|H_1| = \left| \int_{t'}^{t} d\tau \int_{t_j}^{\tau} [f''(\zeta) - y''_j] \, d\zeta \right| \leqslant C\omega_2(\delta) \, \delta \, |t-t'|,$$

$$|H_2| \leqslant C \frac{\omega_2(\delta)}{|\Delta t_j|} \left| \int_{t'}^{t} (s - s_j)(4s_{j+1} - s_j - 3s) \, ds \right| \leqslant C\omega_2(\delta) \, \delta \, |t-t'|$$

and similar estimates for $|H_3|$ and $|H_4|$. So we obtain

$$|e_\Delta(t)-e_\Delta(t')|\leqslant C\omega_2(\delta)\,\delta\,|t-t'|\leqslant C\omega_2(\delta)\,\delta^{2-\varepsilon}|t-t'|^\varepsilon,$$

which may be verified to be valid also if t, t' belong to different L_j's. Thus,

$$M_\varepsilon(e_\Delta)\leqslant C\omega_2(\delta)\,\delta^{2-\varepsilon}.$$

Together with (4.2), we have, by (1.7),

$$\|T_L e_\Delta\|_\infty\leqslant C_\varepsilon\omega_2(\delta)\,\delta^{2-\varepsilon}. \tag{4.7}$$

In order to get a better estimate $\|T_L e'_\Delta\|_\infty$, we differentiate (4.6):

$$\begin{aligned}S'_j(t)=y'_j+y''_j(t-t_j)&-\left[\Delta y_j-y'_j\,\Delta t_j-\frac{1}{2}\,y''_j\,\Delta t_j^2\right]\frac{(t-t_j)(3t+t_j+4t_{j+1})}{\Delta t_j^3}\\&-\left[\Delta y_j-y'_{j+1}\,\Delta t_j+\frac{1}{2}\,y''_{j+1}\,\Delta t_j^2\right]\frac{(t-t_j)(3t-t_j-2t_{j+1})}{\Delta t_j^3}\\&+\frac{y''_{j+1}-y''_j}{2}\,\frac{(t-t_j)(3t-t_j-2t_{j+1})}{\Delta t_j}.\end{aligned}$$

Hence, if $t, t'\in L_j$, we have

$$\begin{aligned}e'_\Delta(t)-e'_\Delta(t')=f'(t)-f'(t')-y''_j(t-t')&-\left[\Delta y_j-y'_j\,\Delta t_j-\frac{1}{2}\,y''_j\,\Delta t_j^2\right]\\&\times\frac{1}{\Delta t_j^3}\int_{t'}^{t}\frac{d}{d\tau}(\tau-t_j)(3\tau+t_j-4t_{j+1})\,d\tau\\&-\left[\Delta y_j-y'_{j+1}\,\Delta t_j+\frac{1}{2}\,y''_{j+1}\,\Delta t_j^2\right]\\&\times\frac{1}{\Delta t_j^3}\int_{t'}^{t}\frac{d}{d\tau}(\tau-t_j)(3\tau-t_j-2t_{j+1})\,d\tau\\&+\frac{y''_{j+1}-y''_j}{2\Delta t_j}\int_{t'}^{t}\frac{d}{d\tau}(\tau-t_j)(3\tau-t_j-2t_{j+1})\,d\tau.\end{aligned}$$

Using the same reasoning as before, it is easily seen

$$|e'_\Delta(t)-e'_\Delta(t')|\leqslant C\omega_2(\delta)\,\delta^{1-\varepsilon}|t-t'|^\varepsilon,$$

which is valid also for t, t' belonging to different L_j's. Thus,

$$M_\varepsilon(e'_\Delta)\leqslant C\omega_2(\delta)\,\delta^{1-\varepsilon},$$

and thereby, by (4.2), we obtain

$$\| T_L e'_\Delta \|_\infty \leqslant C_\varepsilon \omega_2(\delta)\, \delta^{1-\varepsilon}. \tag{4.8}$$

If $f(t) \in C^3$, after rewriting $S_j(t)$ as [9]

$$\begin{aligned} S_j(t) &= y_j + y'_j(t-t_j) + \frac{y''_j}{2}(t-t_j)^2 + \frac{y'''_j}{6}(t-t_j)^3 \\ &\quad - \left[\Delta y_j - y'_j\,\Delta t_j - \frac{y''_j}{2}\Delta t_j^2 - \frac{y'''_j}{6}\Delta t_j^3\right] \frac{(t-t_j)^2\,(t+t_j-2t_{j+1})}{\Delta t_j^3} \\ &\quad - \left[\Delta y_j - y'_{j+1}\,\Delta t_j + \frac{y''_{j+1}}{2}\Delta t_j^2 - \frac{y'''_{j+1}}{6}\Delta t_j^3\right] \frac{(t-t_j)^2\,(t-t_{j+1})}{\Delta t_j^3} \\ &\quad + [\,y''_{j+1} - y''_j - y'''_j\,\Delta t_j]\,\frac{(t-t_j)^2\,(t-t_{j+1})}{2\Delta t_j} \\ &\quad - \frac{y'''_{j+1} - y'''_j}{6}(t-t_j)^2\,(t-t_{j+1}) \end{aligned}$$

and proceeding as before, we may get

$$M_\varepsilon(e_\Delta) \leqslant C\omega_3(\delta)\,\delta^{3-\varepsilon}$$

and hence

$$\| T_L e_\Delta \|_\infty \leqslant C_\varepsilon \omega_3(\delta)\, \delta^{3-\varepsilon}. \tag{4.9}$$

In the same manner, we may also get

$$\| T_L e'_\Delta \|_\infty \leqslant C_\varepsilon \omega_3(\delta)\, \delta^{2-\varepsilon}. \tag{4.10}$$

We also note that, if L is open, we have in this case

$$S_\Delta(a) = f(a), \qquad S_\Delta(b) = f(b), \qquad S'_\Delta(a) = f'(a), \qquad S'_\Delta(b) = f'(b),$$

so that all the above results remain true.

Therefore, we obtain

THEOREM 5. *Let $S_\Delta(t)$ be the cubic interpolating spline of deficiency* 2. *If $f(t) \in C^r$, then*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \omega_r(\delta)\, \delta^{r-\varepsilon}, \qquad r = 1, 2, 3, \tag{4.11}$$

$$|T_L f' - T_L S'_\Delta| \leqslant C_\varepsilon \omega_r(\delta)\, \delta^{r-1-\varepsilon}, \qquad r = 2, 3, \tag{4.12}$$

and if $f'(t) \in H^\alpha$ $(0 < \alpha \leqslant 1)$, *then*

$$|T_L f' - T_L S'_\Delta| \leqslant C_\varepsilon K_\Delta^\varepsilon \delta^{\alpha-\varepsilon}, \tag{4.13}$$

whether L *is a smooth contour or an open smooth arc.*

5. The Modified Cubic Interpolating Splines of Deficiency 2

When $f(t)$ does not possess any derivative but only $\in H^\alpha$, in order to approximate $T_L f$ by smooth functions, we may use the modified cubic interpolating splines introduced in [9].

Suppose L is closed and $S_\Delta(t)$ is the linear spline as in Section 2. By taking two points t'_j and t''_j, respectively, on each L_{j-1} and L_j such that

$$|\widehat{t'_j t_j}| = |\widehat{t_j t''_j}| = \lambda \min(\Delta s_{j-1}, \Delta s_j), \qquad \lambda \leqslant \tfrac{1}{2}.$$

We interpolate $S_\Delta(t)$ cubically on each $L'_j = \widehat{t'_j t''_j}$ with the values and the first derivatives of $S_\Delta(t)$ at t'_j, t''_j and get $S_j^*(t)$. Then we defined [9]

$$\begin{aligned} S_\Delta^*(t) &= S_j^*(t), \qquad \text{when} \quad t \in L'_j, \\ &= S_\Delta(t), \qquad \text{otherwise}, \end{aligned}$$

and proved that, if $f(t) \in H^\alpha$ $(0 < \alpha \leqslant 1)$,

$$|e_\Delta^*(t)| = |f(t) - S_\Delta^*(t)| \leqslant C\delta^\alpha. \tag{5.1}$$

Let us estimate $M_\varepsilon(e_\Delta^*)$. On each arc $\widehat{t''_j t'_{j+1}}$, $S_\Delta^*(t) = S_\Delta(t)$, so, if t, t' belong to it, by (2.3)$'$,

$$|e_\Delta^*(t) - e_\Delta^*(t')| \leqslant C\delta^{\alpha-\varepsilon} |t - t'|^\varepsilon. \tag{5.2}$$

If t, t' belong to L'_j, we have, similar to (4.3),

$$\begin{aligned} e_\Delta^*(t) - e_\Delta^*(t') = {} & [S_\Delta(t) - S_\Delta(t') + D_j^*(t - t')] \\ & + \frac{S'_{j-1}(t'_j) - D_j^*}{\Delta t_j'^2} \int_{t'}^{t} \frac{d}{d\tau} (\tau - t'_j)(\tau - t''_j)^2 \, d\tau \\ & + \frac{S'_j(t''_j) - D_j^*}{\Delta t_j'^2} \int_{t'}^{t} \frac{d}{d\tau} (\tau - t'_j)^2 (\tau - t''_j) \, d\tau \\ = {} & G_1 + G_2 + G_3, \end{aligned}$$

where

$$D_j^* = \frac{S_j(t_j'') - S_{j-1}(t_j')}{\Delta t_j'}, \qquad \Delta t_j' = t_j'' - t_j'.$$

Noting that $S_j(t_j) = S_{j-1}(t_j)$ and

$$S_j(t_j'') - S_j(t_j) = D_j(t_j'' - t_j),$$

we have

$$\begin{aligned} |D_j^*| &\leqslant \frac{1}{|\Delta t_j'|} [|S_j(t_j'') - S_j(t_j)| + |S_{j-1}(t_j) - S_{j-1}(t_j')|] \\ &\leqslant \frac{1}{|\Delta t_j'|} (|D_j|\,|t_j'' - t_j| + |D_{j-1}|\,|t_j' - t_j|) \\ &\leqslant C \left[\frac{1}{|\Delta t_j|^{1-\alpha}} + \frac{1}{|\Delta t_{j-1}|^{1-\alpha}}\right], \end{aligned}$$

and thereby

$$|G_1| \leqslant C\,|t - t'|^\alpha + C\left[\frac{1}{|\Delta t_j|^{1-\alpha}} + \frac{1}{|\Delta t_{j-1}^{1-\alpha}|}\right] |t - t'| \leqslant C\,|t - t'|^\alpha.$$

On the other hand,

$$|S_{j-1}'(t_j') - D_j^*| = |D_{j-1} - D_j| \frac{|t_j'' - t_j|}{|\Delta t_j'|} \leqslant C\left[\frac{1}{|\Delta t_{j-1}|^{1-\alpha}} + \frac{1}{|\Delta t_j|^{1-\alpha}}\right],$$

while the modulus of the integral in G_2 is not greater than $C\,|\widehat{t_j' t_j''}|^2\,|\widehat{tt'}|$, and $|\widehat{t_j' t_j''}|/|\Delta t_j'| \leqslant C$, so that we have

$$|G_2| \leqslant C\,|t - t'|^\alpha$$

and a similar estimate for $|G_3|$. Therefore,

$$|e_\Delta^*(t) - e_\Delta^*(t')| \leqslant C\,|t - t'|^\alpha \leqslant C\delta^{\alpha-\varepsilon}\,|t - t'|^\varepsilon. \tag{5.3}$$

If t, t' are situated in neither of the above two cases, we may then always find two points $\tau_1, \tau_2 \in \{t_j', t_j''\}$ (maybe $\tau_1 = \tau_2$) on the shorter arc $\widehat{tt'}$ such that $\widehat{t\tau_1}$ is a sub-arc of some $\widehat{t_j' t_j''}$ or $\widehat{t_j'' t_{j+1}'}$ and so is $\widehat{\tau_2 t'}$. Then proceeding as before, we may verify (5.3) remains valid. Thus

$$M_\varepsilon(e_\Delta^*) \leqslant C\delta^{\alpha-\varepsilon}.$$

Together with (5.1), we have, by (1.7),

$$\| T_L e_\Delta^* \|_\infty \leqslant C_\varepsilon \delta^{\alpha - \varepsilon} \tag{5.4}$$

if $\alpha < 1$, and so also if $\alpha = 1$.

If L is an open smooth arc, we need not modify $S_\Delta(t)$ near the endpoints a and b, so that

$$S_\Delta^*(a) = f(a), \qquad S_\Delta^*(b) = f(b),$$

and therefore (5.4) is also valid in this case.

Hence, we obtain

THEOREM 6. *If L is a smooth curve, closed or not, $f(t) \in H^\alpha$ $(0 < \alpha \leqslant 1)$ and $S_\Delta^*(t)$ is the modified cubic interpolating spline of deficiency 2 described as above, then*

$$| T_L f - T_L S_\Delta^* | \leqslant C_\varepsilon \delta^{\alpha - \varepsilon}.$$

Remark. All the results in this paper are valid when L is a piece-wise smooth curve without cusps, since inequality (1.9) remains true in this case.

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